

Rational BV-algebra in String Topology

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To Micheline Vigué-Poirrier on her 60th birthday

Abstract

Let M be a 1-connected closed manifold and LM be the space of free loops on M . In [3] M. Chas and D. Sullivan defined a structure of BV-algebra on the singular homology of LM , $H_*(LM; \mathbb{K})$. When the field of coefficients is of characteristic zero, we prove that there exists a BV-algebra structure on $HH^*(C^*(M); C^*(M))$ which carries the canonical structure of Gerstenhaber algebra. We construct then an isomorphism of BV-algebras between $HH^*(C^*(M); C^*(M))$ and the shifted $H_{*+m}(LM; \mathbb{K})$. We also prove that the Chas-Sullivan product and the BV-operator behave well with the Hodge decomposition of $H_*(LM)$.

1 Introduction

Chas and Sullivan considered in [3] the free loop space $LM = \text{map}(S^1, M)$ for a smooth orientable closed manifold of dimension m . They use geometric methods to show that the shifted homology $\mathcal{H}_*(LM) := H_{*+m}(LM)$ has the structure of a Batalin-Vilkovisky algebra (BV-algebra for short). Later on Cohen and Jones defined in [5] a ring spectrum structure on the Thom spectrum LM^{-TM} which realizes the Chas-Sullivan product in homology, that is called the loop product. More recently, Gruher and Salvatore proved in [16] that the algebra structure (and thus the BV-algebra structure) on $\mathcal{H}_*(LM)$ is natural with respect to smooth orientation preserving homotopy equivalences.

Assume that the coefficients ring is a field. By a result of Jones, [18] there exists a linear isomorphism $HH_*(C^*(M); C^*(M)) \cong H^*(LM)$, and by duality an isomorphism $H_*(LM) \cong HH^*(C^*(M); C_*(M))$. Here $HH_*(A; Q)$ (respectively $HH^*(A; Q)$) denotes the Hochschild homology (respectively cohomology) of a differential graded algebra A with coefficients in the differential graded A -bimodule Q , $C^*(M)$ denotes the singular cochains algebra and $C_*(M)$ the complex of singular chains. The cap product induces an isomorphism of graded vector spaces (for instance see [11]-Appendix), $HH^*(C^*(M); C_*(M)) \cong HH^{*-m}(C^*(M); C^*(M))$, and therefore an isomorphism

$$\mathcal{H}_*(LM) \cong HH^*(C^*(M); C^*(M)).$$

Since $HH^*(A; A)$ is canonically a Gerstenhaber algebra, for any differential graded algebra A , it is natural to ask:

Question 1. *Does there exist an isomorphism of Gerstenhaber algebras between $\mathcal{H}_*(LM)$ and $HH^*(C^*(M); C^*(M))$?*

Various isomorphisms of graded algebras have been constructed. The first one has been constructed by Cohen and Jones for any field of coefficients in terms of spectra, [5]. Then Merkulov did it for real coefficients [23], [13] using iterated integrals. Finally for rational coefficients, M. Vigué and the two authors have constructed another isomorphism, [12], using the chain coalgebra of the Quillen minimal model of M .

Although $HH^*(A; A)$ does not have, for any differential graded algebra A , a natural structure of BV-algebra extending the canonical Gerstenhaber algebra, a second natural question is:

Question 2. *Does there exist on $HH^*(C^*(M); C^*(M))$ a structure of BV-algebra extending the structure of Gerstenhaber algebra and an isomorphism of BV-algebras between $\mathcal{H}_*(LM)$ and $HH^*(C^*(M); C^*(M))$?*

The main result of this paper furnishes a positive answer to Question 2 and thus to Question 1 when the field of coefficients is assumed of characteristic zero.

Theorem 1. *If M is 1-connected and the field of coefficients has characteristic zero then there exists a BV-structure on $HH^*(C^*(M); C^*(M))$ and an isomorphism of BV-algebras $\mathcal{H}_*(LM) \cong HH^*(C^*(M); C^*(M))$.*

BV-algebra structures on the Hochschild cohomology $HH^*(A; A)$ have been constructed by different authors under some conditions on A . First of all, Tradler and Zeinalian did it when A is the dual of an A_∞ -coalgebra with ∞ -duality (rational coefficients), [29]. This is in particular the case when $A = C^*(M)$, [28]. In [21] Menichi constructed also a BV-structure in the case when A is a symmetric algebra (any coefficients). Let mention also the construction due to Vaintrob of a BV-structure on $HH^*(A; A)$ when A is the group ring with rational coefficients of the fundamental group of an aspherical manifold [30]. Vaintrob gives also an isomorphism of BV-algebras between $\mathcal{H}(LM)$ and $HH^*(A; A)$. This is coherent with our Theorem 1 because in this case $C_*(\Omega M)$ is quasi-isomorphic to A and using [9], we have isomorphisms of Gerstenhaber algebras

$$HH^*(A; A) \cong HH^*(C_*(\Omega M); C_*(\Omega M)) \cong HH^*(C^*(M); C^*(M)).$$

Extending Theorem 1 to finite fields of coefficients would be difficult. For instance Menichi proved in [22] that the algebras $\mathcal{H}(LS^2)$ and $HH^*(H^*(S^2); H^*(S^2))$ are isomorphic as Gerstenhaber algebras but not as BV-algebras for $\mathbb{Z}/2$ -coefficients.

In this paper we work over a field of characteristic zero. We use rational homotopy and we refer to [7] for rational models. We only recall that a morphism in some category of complexes is a *quasi-isomorphism* if it induces an isomorphism in homology. Two objects are *quasi-isomorphic* if they are related by a finite sequence of quasi-isomorphisms.

Now by a recent result of Lambrechts and Stanley, [19] there is a commutative differential graded algebra A satisfying:

1. A is quasi-isomorphic to the differential graded algebra $C^*(M)$.
2. A is connected, finite dimensional and satisfies Poincaré duality in dimension m .
This means there exists a A -linear isomorphism $\theta : A \rightarrow A^\vee$ of degree $-m$ which commutes with the differentials.

We call A a Poincaré duality model for M . This model can replace $C^*(M)$ in the Hochschild cohomology because there is an isomorphism of Gerstenhaber algebra between $HH^*(C^*(M); C^*(M))$ and $HH^*(A; A)$ ([9]).

Denote by μ the multiplication of A . We define the linear map $\mu_A : A \rightarrow A \otimes A$ by the commutative diagram

$$(*) \quad \begin{array}{ccc} A^\vee & \xrightarrow{\mu^\vee} & (A \otimes A)^\vee = A^\vee \otimes A^\vee \\ \theta \uparrow & & \uparrow \theta \otimes \theta \\ A & \xrightarrow{\mu_A} & A \otimes A \end{array}$$

By definition μ_A is a degree m homomorphism of A -bimodules which commutes with the differentials.

Let $\mathbf{C}_*(A; A) := (A \otimes T(s\bar{A}), \partial)$ be the Hochschild complex of A with coefficients in A . Here $T(s\bar{A})$ denotes the free coalgebra generated by the graded vector space $s\bar{A}$ with $\bar{A} = \{A^i\}_{i \geq 1}$ and $(s\bar{A})^i = A^{i+1}$. We prove:

Proposition 1.

1. The complex $\mathbf{C}_*(A; A)$ is quasi-isomorphic to $C^*(LM)$.
2. If $\phi : T(s\bar{A}) \rightarrow T(s\bar{A}) \otimes T(s\bar{A})$ denotes the coproduct of $T(s\bar{A})$ then the composition

$$\Phi : A \otimes T(s\bar{A}) \xrightarrow{id \otimes \phi} A \otimes T(s\bar{A}) \otimes T(s\bar{A}) \cong A \otimes_{A \otimes 2} (A \otimes T(s\bar{A}))^{\otimes 2} \xrightarrow{\mu_A \otimes id} A^{\otimes 2} \otimes_{A \otimes 2} (A \otimes T(s\bar{A}))^{\otimes 2}$$

is a linear map of degree m which commutes with the differentials.

3. The isomorphism $HH_*(A; A)^{*+m} \cong H^*(LM)$ transfers the map induced by Φ to the dual to the Chas-Sullivan product.
4. The duality isomorphism $HH_*(A; A)^\vee \cong HH^*(A; A^\vee) \xrightarrow{(\theta)} HH^{*-m}(A; A)$ transfers the map induced by Φ on $HH_*(A; A)$ to the Gerstenhaber product on $HH^*(A; A)$.

Denote by $\Delta : \mathcal{H}_*(LM) \rightarrow \mathcal{H}_{*+1}(LM)$ and $\Delta' : \mathcal{H}^*(LM) \rightarrow \mathcal{H}^{*-1}(LM)$ the morphisms induced by the canonical action of S^1 on LM . As proved by Chas and Sullivan the morphism Δ together with the loop product equip the loop space homology $H_*(M)$ with a BV-structure. We prove:

Proposition 2. The Connes' boundary $B : \mathbf{C}_*(A; A) \rightarrow \mathbf{C}_{*-1}(A; A)$ induces a map in homology corresponding to the operator Δ' via the isomorphism $HH_*(A; A) \cong H^*(LM)$.

On the other hand, the duality isomorphism $HH_*(A; A)^\vee \cong HH^*(A; A^\vee) \xrightarrow{(\theta)} HH^*(A; A)$ transfers the linear dual of the Connes' operator B to an operator on $HH^*(A; A)$ that gives to the Hochschild cohomology a BV-structure extending the Gerstenhaber algebra structure ([21],[28]) and $HH^*(A; A) \cong HH_*(C^*(M); C^*(M))$ as Gerstenhaber algebras [9]. This fact combined with Proposition 1 and 2 gives Theorem 1.

Since the field of coefficients is of characteristic zero, the homology of LM admits a Hodge decomposition, $\mathcal{H}_*(LM) = \oplus_{r \geq 0} \mathcal{H}_*^{[r]}(LM)$, [15], [20, Theorem 4.5.10]. We prove that this decomposition behaves well with respect to the loop product \bullet and the BV-operator Δ defined by Chas-Sullivan.

Theorem 2. With the above notations, we have

$$\bullet \quad \mathcal{H}_*^{[r]}(LM) \otimes \mathcal{H}_*^{[s]}(LM) \xrightarrow{\bullet} \mathcal{H}_*^{[\leq r+s]}(LM),$$

- $\Delta : \mathcal{H}_*^{[r]}(LM) \rightarrow \mathcal{H}_*^{[r+1]}(LM)$.

By definition $\mathcal{H}_*^{[0]}(LM)$ is the image of $H_{*-m}(M)$ by the homomorphism induced in homology by the canonical section $M \rightarrow LM$. It has been proved in [10] that if $\text{aut}M$ denotes the monoid of (unbased) self-equivalences of M then there exist an isomorphism of graded algebras $\mathcal{H}_*^{[1]}(LM) \cong H_{*-m}(M) \otimes \pi_*(\Omega \text{aut}M)$. For any $r \geq 0$, a description of $\mathcal{H}_*^{[r]}(LM)$ can be obtained, using a Lie model (L, d) of M , as proved in the last result.

Proposition 3. *The graded vector space $\mathcal{H}_*^{[r]}(LM)$ is isomorphic to $\text{Tor}^{UL}(\mathbf{k}, \Gamma^r(L))$ where $\Gamma^r(L)$ is the sub UL -module of UL for the adjoint representation that is the image of $\wedge^r L$ by the classical Poincaré-Birkhoff-Witt isomorphism of coalgebras $\wedge L \rightarrow UL$.*

The text is organized as follows. Notation and definitions are made precise in sections 2 and 3. Proposition 1 is proved in Sections 4, Proposition 2 is proved in section 5. Theorem 2 and Proposition 3 are proved in the last section.

2 Hochschild homology and cohomology

2.1 Bar construction

Let A be a differential graded augmented cochain algebra and let P (respectively N) be a differential graded right (respectively left) A -module, $A = \{A^i\}_{i \geq 0}$, $P = \{P^j\}_{j \in \mathbb{Z}}$, $N = \{N^j\}_{j \in \mathbb{Z}}$ and $\bar{A} = \ker(\varepsilon : A \rightarrow \mathbf{k})$. The *two-sided (normalized) bar construction*,

$$\mathcal{B}(P; A; N) = P \otimes T(s\bar{A}) \otimes N, \quad \mathcal{B}_k(P; A; N) = P \otimes T^k(s\bar{A}) \otimes N,$$

is the cochain complex defined as follows: For $k \geq 1$, a generic element $p[a_1|a_2|\dots|a_k]n$ in $\mathcal{B}_k(P; A; N)$ has (upper) degree $|p| + |n| + \sum_{i=1}^k(|sa_i|)$. If $k = 0$, we write $p[]n = p \otimes 1 \otimes n \in P \otimes T^0(s\bar{A}) \otimes N$. The differential $d = d_0 + d_1$ is defined by:

$$\begin{aligned} \mathcal{B}_k(P; A; N)^l &\xrightarrow{d_0} \mathcal{B}_k(P; A; N)^{l+1}, \quad \mathcal{B}_k(P; A; N)^l \xrightarrow{d_1} \mathcal{B}_{k-1}(P; A; N)^{l+1} \\ d_0(p[a_1|a_2|\dots|a_k]n) &= d(p)[a_1|a_2|\dots|a_k]n - \sum_{i=1}^k (-1)^{\epsilon_i} p[a_1|a_2|\dots|d(a_i)|\dots|a_k]n \\ &\quad + (-1)^{\epsilon_{k+1}} p[a_1|a_2|\dots|a_k]d(n) \\ d_1(p[a_1|a_2|\dots|a_k]n) &= (-1)^{|p|} p a_1[a_2|\dots|a_k]n + \sum_{i=2}^k (-1)^{\epsilon_i} p[a_1|a_2|\dots|a_{i-1}a_i|\dots|a_k]n \\ &\quad - (-1)^{\epsilon_k} p[a_1|a_2|\dots|a_{k-1}]a_k n \end{aligned}$$

Here $\epsilon_i = |p| + \sum_{j < i}(|sa_j|)$.

In particular, considering \mathbf{k} as a trivial A -bimodule we obtain the complex $\mathcal{B}A = \mathcal{B}(\mathbf{k}; A; \mathbf{k})$ which is a differential graded coalgebra whose comultiplication is defined by

$$\phi([a_1|\dots|a_r]) = \sum_{i=0}^r [a_1|\dots|a_i] \otimes [a_{i+1}|\dots|a_r].$$

Recall that a differential A -module N is called *semifree* if N is the union of an increasing sequence of sub-modules $N(i)$, $i \geq 0$, such that each $N(i)/N(i-1)$ is an R -free module on a basis of cycles ([7]). Then,

Lemma 1. [7, Lemma 4.3] *The canonical map $\varphi : \mathbb{B}(A; A; A) \rightarrow A$ defined by $\varphi[\cdot] = 1$ and $\varphi([a_1 | \cdots | a_k]) = 0$ if $k > 0$, is a semifree resolution of A as an A -bimodule.*

2.2 Hochschild complexes

Let us denote by $A^e = A \otimes A^{op}$ the envelopping algebra of A .

If P is a differential graded (right) A -bimodule then the cochain complex

$$\mathbf{C}_*(P; A) := (P \otimes T(s\bar{A}), \partial) \stackrel{def}{\cong} P \otimes_{A^e} \mathbb{B}(A; A; A)$$

is called the *Hochschild chain complex of A with coefficients in P* . Its homology is called the *Hochschild homology of A with coefficients in P* and is denoted by $HH_*(A; P)$. When we consider $\mathbf{C}_*(A; A)$ as well as $HH_*(A; A)$, A is supposed equipped with its canonical bimodule structure.

If N is a (left) differential graded A -bimodule then the cochain complex

$$\mathbf{C}^*(A; N) := (\text{Hom}(T(s\bar{A}), N), \delta) \stackrel{def}{\cong} \text{Hom}_{A^e}(\mathbb{B}(A; A; A), N)$$

is called the *Hochschild cochain complex of A with coefficients in the differential graded A -bimodule N* . Its cohomology is called the *Hochschild cohomology of A with coefficients in N* and is denoted by $HH^*(A; N)$. When we consider $\mathbf{C}^*(A; A)$ as well as $HH^*(A; A)$, A is supposed equipped with its canonical bimodule structure.

Let us denote by V^\vee the *graded dual* of the graded vector space $V = \{V^i\}_{i \in \mathbb{Z}}$, i.e. $V^\vee = \{V_i^\vee\}_{i \in \mathbb{Z}}$ with $V_i^\vee := \text{Hom}(V^i, \mathbf{k})$. The canonical isomorphism

$$\text{Hom}(A \otimes_{A^e} \mathbb{B}(A; A; A), \mathbf{k}) \rightarrow \text{Hom}_{A^e}(\mathbb{B}(A; A; A), A^\vee)$$

induces the isomorphism of complexes $\mathbf{C}_*(A; A)^\vee \rightarrow \mathbf{C}^*(A; A^\vee)$.

2.3 The Gerstenhaber algebra on $HH^*(A; A)$

A *Gerstenhaber algebra* is a commutative graded algebra $H = \{H_i\}_{i \in \mathbb{Z}}$ with a bracket

$$H_i \otimes H_j \rightarrow H_{i+j-1}, \quad x \otimes y \mapsto \{x, y\}$$

such that for $a, a', a'' \in H$:

- (a) $\{a, a'\} = (-1)^{(|a|-1)(|a'|-1)} \{a', a\}$
- (b) $\{a, \{a', a''\}\} = \{\{a, a'\}, a''\} + (-1)^{(|a|-1)(|a'|-1)} \{a', \{a, a''\}\}.$

For instance the Hochschild cohomology $HH^*(A; A)$ is a Gerstenhaber algebra [14]. The bracket can be defined by identifying $\mathbf{C}^*(A; A)$ with a differential graded Lie algebra of coderivations ([24], [9, 2.4]).

2.4 BV-algebras and differential graded Poincaré duality algebras.

A Batalin-Vilkovisky algebra (BV-algebra for short) is a commutative graded algebra, H together with a linear map (called a BV-operator)

$$\Delta : H^k \rightarrow H^{k-1}$$

such that

- (1) $\Delta \circ \Delta = 0$
- (2) H is a Gerstenhaber algebra with the bracket defined by

$$\{a, a'\} = (-1)^{|a|} \left(\Delta(aa') - \Delta(a)a' - (-1)^{|a|} ab\Delta(a') \right).$$

A differential graded Poincaré algebra (A, d) is a finite dimensional commutative differential graded algebra together with an isomorphism of differential A -modules, $\theta : A \rightarrow A^\vee$. When (A, d) is a differential graded Poincaré algebra, then $HH^*(A; A)$ is a BV-algebra [21]. The BV-operator Δ is obtained from the Connes' boundary on $\mathbf{C}_*(A; A)$:

$$B : C_n(A; A) \rightarrow C_{n+1}(A; A),$$

$$B(a_0 \otimes [a_1 | \dots | a_n]) = \begin{cases} 0 & \text{si } |a_0| = 0 \\ \sum_{i=0}^n (-1)^{\bar{\epsilon}_i} 1 \otimes [a_i | \dots | a_n | a_0 | a_1 | \dots | a_{i-1}] & \text{si } |a_0| > 0 \end{cases}$$

where

$$\bar{\epsilon}_i = (|sa_0| + |sa_1| + \dots + |sa_{i-1}|)(|sa_i| + \dots + |sa_n|).$$

It is well known that $B^2 = 0$ and $B \circ \partial + \partial \circ B = 0$.

The operator Δ is the image of B^\vee by the duality isomorphism $(HH_{*+m}(A; A))^\vee \cong HH^{*+m}(A; A^\vee) \xrightarrow{(\theta)} HH^*(A; A)$.

3 The Chas-Sullivan algebra structure on $\mathcal{H}_*(LM)$ and its dual

We assume in this section and in the following ones that \mathbf{k} is a field of characteristic zero.

Denote by $p_0 : LM \rightarrow M$ the evaluation map at the base point, and recall that the space LM can be replaced by a smooth manifold ([4], [25]) so that p_0 is a smooth locally trivial fibre bundle ([1], [25]).

The loop product

$$\bullet : H_*(LM)^{\otimes 2} \rightarrow H_{*-m}(LM), \quad x \otimes y \mapsto x \bullet y$$

was first defined by M. Chas and D. Sullivan, [3], by using “transversal geometric chains”. With the loop product, $\mathcal{H}_*(LM) := H_{*+m}(LM)$ is a commutative graded algebra.

It is convenient for our purpose to introduce the *dual of the loop product* $H^*(LM) \rightarrow H^{*+m}(LM^{\times 2})$. Consider the commutative diagram

$$(1) \quad \begin{array}{ccccc} LM^{\times 2} & \xleftarrow{i} & LM \times_M LM & \xrightarrow{\text{Comp}} & LM \\ p_0^{\times 2} \downarrow & & p_0 \downarrow & & \downarrow p_0 \\ M^{\times 2} & \xleftarrow{\Delta} & M & = & M \end{array}$$

where

- Comp denotes composition of free loops,
- the left hand square is a pullback diagram of locally trivial fibrations,
- i is the embedding of the manifold of composable loops into the product $LM \times LM$.

The embeddings Δ and i have both codimension m . Thus, using the Thom-Pontryagin construction we obtain the Gysin maps:

$$\Delta^! : H^k(M) \rightarrow H^{k+m}(M^{\times 2}), \quad i^! : H^k(LM \times_M LM) \rightarrow H^{k+m}(LM^{\times 2}).$$

Thus diagram (1) yields the following diagram:

$$(2) \quad \begin{array}{ccccc} H^{k+m}(LM^{\times 2}) & \xleftarrow{i^!} & H^k(LM \times_M LM) & \xleftarrow{H^k(\text{Comp})} & H^k(LM) \\ H^*(p_0)^{\otimes 2} \uparrow & & H^*(p_0) \uparrow & & \uparrow H^*(p_0) \\ H^{k+m}(M^{\times 2}) & \xleftarrow{\Delta^!} & H^k(M) & = & H^k(M) \end{array}$$

Following ([26], [6]), the *dual of the loop product* is defined by composition of maps on the upper line :

$$i^! \circ H^*(\text{Comp}) : H^*(LM) \rightarrow H^{*+m}(LM^{\times 2}).$$

4 A Hochschild chain model for the dual of the loop product

The composition of free loops $\text{Comp} : LM \times_M LM \rightarrow LM$ is obtained by pullback from the composition of paths $\text{Comp}' : M^I \times_M M^I \rightarrow M^I$ in the following commutative diagram.

$$(Comp) \quad \begin{array}{ccccc} & LM \times_M LM & \xrightarrow{j} & M^I \times_M M^I & \\ & \swarrow \text{Comp} & & \swarrow \text{Comp}' & \\ LM & \xrightarrow{j} & M^I & & \\ \downarrow ev_0=p_0 & & \downarrow ev_0 & & \downarrow (ev_0, ev_1, ev_0) \\ & M & \xrightarrow{(id \times \Delta) \circ \Delta} & M^{\times 3} & \\ & \swarrow \Delta & & \swarrow pr_{13} & \\ M & \xrightarrow{\Delta} & M^{\times 2} & & \end{array}$$

Here Δ denotes the diagonal embedding, j the obvious inclusions, ev_t denotes the evaluation maps at t , and pr_{13} the map defined by $pr_{13}(a, b, c) = (a, c)$.

Let (A, d) be a commutative differential graded algebra quasi-isomorphic to the differential graded algebra $C^*(M)$. A cochain model of the right hand square in diagram $(Comp)$ is given by the diagram

$$(i) \quad \begin{array}{ccc} B(A; A; A) & \xrightarrow{\Psi} & B(A; A; A) \otimes_A B(A; A; A) \\ \uparrow & & \uparrow \\ A^{\otimes 2} & \xrightarrow{\psi} & A^{\otimes 3} \end{array}$$

where Ψ and ψ denote the homomorphism of cochain complexes defined by

$$\Psi(a \otimes [a_1 | \dots | a_k] \otimes a') = \sum_{i=0}^k a \otimes [a_1 | \dots | a_i] \otimes 1 \otimes [a_{i+1} | \dots | a_k] \otimes a', \text{ and } \psi(a \otimes a') = a \otimes 1 \otimes a'.$$

The multiplication $\mu : A \otimes A \rightarrow A$ makes A a A -bimodule, and is a model for the diagonal Δ . We consider now the diagram obtained by tensoring diagram (\dagger) by A over $A^{\otimes 2}$.

$$\begin{array}{ccc}
 A \otimes_{A^{\otimes 2}} \mathcal{B}(A, A, A) & \xrightarrow{id \otimes \Psi} & A \otimes_{A^{\otimes 2}} \mathcal{B}(A; A; A) \otimes_A \mathcal{B}(A; A; A) \\
 \uparrow & & \uparrow \\
 A \otimes_{A^{\otimes 2}} A^{\otimes 2} & \xrightarrow{id \otimes \psi} & A \otimes_{A^{\otimes 2}} A^{\otimes 3}
 \end{array}
 \quad (\ddagger)$$

Since $\mathcal{B}(A; A; A)$ is a semifree model of A as A -bimodule, we deduce from [7] that this diagram is a cochain model of the left hand square. Thus we have proved:

Lemma 2. *The cochain complex $\mathbf{C}_*(A; A)$ is a cochain model of LM , and we have an isomorphism of graded vector spaces*

$$HH_*(A, A) \cong H^*(LM).$$

Moreover, if ϕ denotes the the coproduct of the coalgebra $T(s\bar{A})$ then the composition Φ ,

$$\mathbf{C}_*(A; A) \cong A \otimes T(s\bar{A}) \xrightarrow{id \otimes \phi} A \otimes T(s\bar{A}) \otimes T(s\bar{A}) = \mathbf{C}_*(A; A) \otimes_A \mathbf{C}_*(A; A),$$

is model of the composition of free loops.

Let (A, d) be a differential graded Poincaré duality model of M , as defined in the Introduction.

Recall now that the Gysin map $\Delta^!$ of the diagonal embedding $\Delta : M \rightarrow M \times M$ is the Poincaré dual of the homomorphism $H_*(\Delta)$. This means that the following diagram is commutative

$$\begin{array}{ccc}
 H_*(M) & \xrightarrow{H_*(\Delta)} & H_*(M \times M) \\
 \uparrow -\cap[M] \cong & & \cong \uparrow -\cap[M \times M] \\
 H^*(M) & \xrightarrow{\Delta^!} & H^*(M \times M)
 \end{array}$$

Thus the linear map of degree $\mu_A = A \rightarrow A \otimes A$ defined in the introduction (Diagram $(*)$) is a cochain model for $\Delta^!$.

Remark, [24], that we can choose the pullback of a tubular neighborhood of the diagonal embedding Δ as a tubular neighborhood of the embedding $i : LM \times_L M \rightarrow LM \times LM$. We deduce that

Lemma 3. *The linear map of degree m*

$$C_*(A; A) \otimes_A C_*(A, A) \cong A \otimes_{A^{\otimes 2}} C_*(A; A)^{\otimes 2} \xrightarrow{\mu_A \otimes id} C_*(A; A)^{\otimes 2}$$

commutes with the differential and induces $i^!$ in homology.

The next Lemma follows from standard computation. In combination with Lemma 2 and 3, it gives the proof of Proposition 1 of the Introduction.

Lemma 4. *The duality isomorphism $(HH_{*+m}(A; A))^\vee \cong HH^{*+m}(A; A^\vee) \stackrel{(\theta)}{\cong} HH^*(A; A)$ transfers the map induced by Φ on $HH_*(A; A)$ to the Gerstenhaber product on $HH^*(A; A)$.*

Remark. By putting $F_p := A \otimes (T(s\bar{A}))^{\leq p}$, for $p \geq 0$, we define a filtration

$$A \otimes T(s\bar{A}) \supset \dots \supset F_p \supset F_{p-1} \supset \dots \supset A = F_0$$

such that $\partial F_p \subset F_p$ and $\Phi(F_p) \subset \oplus_{k+\ell=p} F_k \otimes F_\ell$. The resulting spectral sequence

$$E_2^{p,q} = H^q(M) \otimes H^p(\Omega M) \implies H^{p+q}(LM)$$

is the multiplicative Serre spectral sequence for the fibration $p_0 : LM \rightarrow M$. It dualizes into a spectral sequence of algebras

$$H_{q+m}(M) \otimes H_p(\Omega M) \implies \mathbb{H}_{p+q}(LM).$$

We recover in this way over \mathbb{Q} the spectral sequence defined previously by Cohen, Jones and Yan [6].

5 The canonical circle action on LM

Let $\rho : S^1 \times LM \rightarrow LM$ be the canonical action of the circle on the space LM . The action ρ induces an operator $\Delta : \mathbb{H}_*(LM) \rightarrow \mathbb{H}_{*+1}(LM)$. The loop product together with Δ gives to $\mathbb{H}_*(LM)$ a BV-structure [3].

Denote by $\mathfrak{M}_M = (\wedge V, d)$ a (non necessary minimal) Sullivan model for M [8, §-12]. We put $sV = \bar{V}$ and denote by S the derivation of $\wedge V \otimes \wedge \bar{V}$ defined by $S(v) = \bar{v}$ and $S(\bar{v}) = 0$ for $v \in V$ and $\bar{v} \in \bar{V}$. Then a Sullivan model for LM is given by the commutative differential graded algebra $(\wedge V \otimes \wedge \bar{V}, \bar{d})$ where $\bar{d}(\bar{v}) = -S(dv)$ [27]. Moreover in [2] Burghlelea and Vigué prove that a Sullivan model of the action $\rho : S^1 \times LM \rightarrow LM$ is given by

$$\begin{aligned} \mathfrak{M}_\rho : (\wedge V \otimes \wedge \bar{V}, \bar{d}) &\rightarrow (\wedge u, 0) \otimes (\wedge V \otimes \wedge \bar{V}, \bar{d}), \quad |u| = 1) \\ \mathfrak{M}_\rho(\alpha) &= 1 \otimes \alpha + u \otimes S(\alpha), \quad \alpha \in \wedge V \otimes \wedge \bar{V} \end{aligned}$$

In particular the map induced in cohomology by the action of S^1 on LM is given by the derivation $S : H^*(\wedge V \otimes \wedge \bar{V}) \rightarrow H^{*-1}(\wedge V \otimes \wedge \bar{V})$. Denote now by B the Connes' boundary on $\mathbf{C}^*(\mathfrak{M}_M; \mathfrak{M}_M) = \wedge V \otimes T(s\bar{\wedge} \bar{V})$. Vigué proved the following Lemma in [31, Theorem 2.4].

Lemma 5. *The morphism $f : \mathbf{C}^*(\mathfrak{M}_M; \mathfrak{M}_M) \rightarrow (\mathfrak{M}_M \otimes \wedge \bar{V})$ defined by*

$$f(a \otimes [a_1 | \cdots | a_n]) = \frac{1}{n!} a S(a_1) \cdots S(a_n)$$

is a quasi-isomorphism of complexes and $f \circ B = S \circ f$.

This, combined with lemma 4, implies directly Proposition 2 of the Introduction.

6 The Hodge decomposition

With the notation of the previous sections, let $(\wedge V \otimes \wedge \bar{V}, \bar{d})$ be a Sullivan model for LM . Denote by $G^p = \wedge V \otimes \wedge^p \bar{V}$ the subvector space generated by the words of length p in \bar{V} . The differential \bar{d} satisfies $\bar{d}(G^p) \subset G^p$. Thus we put

$$H_{[p]}^n(LM) := H^n(G^p).$$

This decomposition has been considered by many authors (see for instance [20], [32]). It induces by duality a Hodge decomposition on $H_*(LM)$. We are now ready to prove Theorem 2 of the Introduction.

Proof of Theorem 2. Recall that the differential d in $\mathbf{C}^*(\mathfrak{M}_M; \mathfrak{M}_M)$ decomposes into $d = d_0 + d_1$ with $d_0(\mathfrak{M}_M \otimes T^p(s\overline{\wedge V})) \subset \mathfrak{M}_M \otimes T^p(s\overline{\wedge V})$, and $d_1(\mathfrak{M}_M \otimes T^p(s\overline{\wedge V})) \subset \mathfrak{M}_M \otimes T^{p-1}(s\overline{\wedge V})$.

We consider the quasi-isomorphism $f : \mathbf{C}^*(\mathfrak{M}_M; \mathfrak{M}_M) \rightarrow (\mathfrak{M}_M \otimes \wedge \bar{V}, \bar{d})$ defined in Lemma 5. If we apply Lemma 5, when $d = 0$ in $\wedge V$, we deduce that $\text{Ker } f$ is d_1 -acyclic. Denote $K^{(p)} = \text{Ker } f \cap (\mathfrak{M}_M \otimes T^p(s\overline{\wedge V}))$. Then

Lemma 6.

1. If $\omega \in K^{(p)} \cap \text{Ker } \partial$ then there exists $\omega' \in \oplus_{r \geq p+1} K^{(r)}$ such that $\partial \omega' = \omega$.
2. f induces a surjective map $(\mathfrak{M}_M \otimes T^{\geq p}(s\overline{\wedge V})) \cap \text{Ker } \partial \twoheadrightarrow (\mathfrak{M}_M \otimes \wedge^p sV) \cap \text{Ker } \bar{d}$.

Proof. If $\omega \in K^{(p)} \cap \text{Ker } \partial$ then $\omega = \partial(u + v)$ with $u \in K^{(p)}$ and $v \in K^{(\geq p+1)}$. Since $\partial_1 u = 0$ we have $u = \partial \beta_1$ some $\beta \in K^{(p+1)}$ and thus $\omega - d\beta_1 \in K^{(\geq p+1)}$. An induction on $n \geq 1$ we prove that there exists $\beta_n \in K^{(p+n)}$ such that $\omega - d\beta_n \in K^{(p+n)}$. Since $\wedge V$ is 1-connected $(\mathfrak{M}_M \otimes T^{p+n}(s\overline{\wedge V}))^{|\omega|} = 0$ for some integer n_0 . We put $\omega' = \beta_{n_0}$.

In order to prove the second statement, we consider a \bar{d} -cocycle $\alpha \in \mathfrak{M}_M \otimes \wedge^p sV$ and we write $\alpha = f(\omega)$ for some $\omega \in \mathfrak{M}_M \otimes T^p(s\overline{\wedge V})$. It follows from the definition of f that $\partial \omega \in K^{(p-1)}$. Thus, by the first statement, $\partial \omega = \partial \omega'$ some $\omega' \in K^{(\geq p)}$. Then $\varpi = \omega - \omega'$ is ∂ -cocycle of $K^{\geq p}$ such that $f(\varpi) = \alpha$. □

To end the proof, let $\alpha \in H_{[n]}^*(LM)$, then by Lemma 6, α is the class of $f(\beta)$ where $\beta \in \mathfrak{M}_M \otimes T^{\geq n}(s\overline{\wedge V})$. Therefore $\Phi(\beta) \in \oplus_{i+j \geq n} (\mathfrak{M}_M \otimes T^i(s\overline{\wedge V})) \otimes (\mathfrak{M}_M \otimes T^j(s\overline{\wedge V}))$ (see Lemma 2). Now since $f(\mathfrak{M}_M \otimes T^p(s\overline{\wedge V})) \subset \mathfrak{M}_M \otimes \wedge^p sV$,

$$[\Phi(\alpha)] \in \oplus_{i+j \geq n} H_{[i]}^*(LM) \otimes H_{[j]}^*(LM).$$

□

Now, as announced in the Introduction (Proposition 3) there is an other interpretation of $H_{[p]}^n(LM)$ in terms of the cohomology of a differential graded Lie algebra.

Let L be a differential graded algebra L such that the cochain algebra $\mathcal{C}^*(L)$ is a Sullivan model of M , [8, p.322]. In particular, the homology of the enveloping universal algebra of L , denoted UL is a Hopf algebra isomorphic to $H_*(\Omega M)$. We consider the cochain complex $\mathcal{C}^*(L; UL_a^\vee)$ of L with coefficients in UL^\vee considered as an L -module for the adjoint representation. We have shown ([12, Lemma 4]) that the natural inclusion $\mathcal{C}^*(L) \hookrightarrow \mathcal{C}^*(L; UL_a^\vee)$ is a relative Sullivan model of the fibration $p_0 : LM \rightarrow M$. Write $\mathcal{C}^*(L) = (\wedge V, d)$, then $V = (sL)^\vee$ and $\bar{V} = L^\vee$. There is also (Poincaré-Birkhoff-Witt Theorem) an isomorphism of graded coalgebras, [8, Proposition 21.2]:

$$\gamma : \bigwedge L \rightarrow UL, \quad x_1 \wedge \dots \wedge x_k \mapsto \sum_{\sigma \in \mathfrak{S}_k} \epsilon_\sigma x_{\sigma(1)} \dots x_{\sigma(k)}.$$

If we put $\Gamma^p = \gamma(\wedge^p V)$ we obtain the following isomorphisms of cochain complexes

$$(\bigwedge V \otimes \bigwedge \bar{V}, \bar{d}) \cong \mathcal{C}^*(L; UL_a^\vee), \quad G^p \cong \mathcal{C}^*(L; (\Gamma^p)^\vee)$$

which in turn induce the isomorphisms:

$$H^*(LM) \cong \text{Ext}_{UL}(\mathbf{k}, UL_a^\vee), \quad H_{[p]}^*(LM) \cong \text{Ext}_{UL}(\mathbf{k}, \Gamma^p(L)^\vee).$$

and by duality,

$$\mathcal{H}_*(LM) \cong \mathrm{Tor}^{UL}(\mathbb{K}, UL_a), \quad \mathcal{H}_*^{[p]}(LM) \cong \mathrm{Tor}^{UL}(\mathbb{K}, \Gamma^p).$$

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